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LINEAR GROWTH HARMONIC FUNCTIONS ON A COMPLETE MANIFOLD

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1. Introduction

In [3], Yau proved that on a complete manifold with nonnegative Ricci curvature, there does not exist any nonconstant positive harmonic function which is defined on the entire manifold. In fact, his argument also showed that the space of sublinear growth harmonic functions on such a manifold consists of constant functions only. More precisely, if M is a complete manifold with nonnegative Ricci curvature and f is a harmonic function on M which satisfies

$$|f|(y) \leq o(r(y))$$

for some distant function r to a fixed point, then f must be identically constant. This led Yau to ask the following questions:

(1) For each integer p, is the space of harmonic functions on a manifold with nonnegative Ricci curvature satisfying

$$|f|(y) \leq O(r^p(y))$$

finite dimensional?

(2) If so, is the dimension less than or equal to the dimension of the space of homogeneous harmonic polynomials of degree less than or equal to p in R^n for $n = \dim M$?

The purpose of this article is to show that when $p = 1$ the answers for the above questions are affirmative. In fact, we will prove that the dimension can be estimated in terms of the order of the volume growth. Together with the volume comparison theorem, this implies that the space of linear growth harmonic functions is of dimension at most n.

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Theorem. *Let M be an n-dimensional Riemannian manifold with nonnegative Ricci curvature. Assume that the volume of the geodesic balls of*

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radius r centered at some point $x \in M$ satisfies

$$V(B_x(r)) = O(r^k).$$

Then the dimension of the space of linear growth harmonic functions on M must be less than or equal to k. In particular, using Yau's theorem, the space of harmonic functions on M satisfying

$$|f|(y) \leq O(r(y))$$

must have dimension less than or equal to $k+1$.

In view of the above theorem, perhaps it is reasonable to ask that if the manifold satisfies the assumption of the theorem, does the space of all harmonic functions on M which grows like

$$|f|(y) \leq O(r^p(y))$$

have dimension less than that of R^k?

2. Slow volume growth

Before we prove the theorem, we would like to remark that the situation simplifies substantially if the volume growth of M is slow, that is

$$\int_1^\infty V(B_x(\sqrt{t}))^{-1}\, dt = \infty.$$

In this case, any linear growth harmonic function must have parallel gradient, and hence the manifold splits into $M = R \times N$, where N is a complete manifold with nonnegative Ricci curvature. More precisely, we have the following:

Proposition. *Let M be a complete manifold with nonnegative Ricci curvature. Suppose there exists a point $x \in M$ such that the volume of the geodesic ball, $V(B_x(r))$, centered at x with radius r satisfies*

$$\int_1^\infty V(B_x(\sqrt{t}))^{-1}\, dt = \infty.$$

If M admits s linearly independent nonconstant linear growth harmonic functions $\{f_i\}$, then M must be isometrically the product $M = R^s \times N$, where N is a complete manifold with nonnegative Ricci curvature. Moreover the harmonic functions f_i must be linear functions of R^s. In particular, the dimension of the space of nonconstant linear growth harmonic functions must be at most 2.

Proof. Let f be a nonconstant linear growth harmonic function. By the gradient estimate of Yau ([3], [1]) the function $|\nabla f|(y)$ is a bounded function on M. Bochner's formula also implies that $|\nabla f|$ is subharmonic on M in the sense of distribution. On the other hand, it is known (see Proposition 2 of

[2]) that if the volume growth of M is small then there does not exist any nonconstant bounded subharmonic function. Hence $|\nabla f|$ must be identically constant. Bochner's formula then implies that the vector field ∇f must be parallel which gives us the splitting for the universal covering $M' = R \times N$ of M. However, since f is defined on M, it is invariant under the action of the deck transformations so M itself must split. If g is another linear growth harmonic function on M which is linearly independent to f, then we may assume that at a point x, the vectors ∇f and ∇g are perpendicular. By the fact that they are both parallel, they must remain to be perpendicular everywhere. Hence g must be a function of N alone. We now argue that N splits as a product $R \times N'$. Inductively, we observe that M must split into $R^s \times N$ if there are s linearly independent nonconstant linearly growth harmonic functions. However, this will violate the assumption on the volume growth unless $s \leq 2$. In fact, if $s = 2$ then N must be a compact manifold with nonnegative Ricci curvature.

3. Fast volume growth

In view of the above proposition, we may assume from here on that M has fast volume growth, that is

$$\int_1^\infty V(B_x(\sqrt{t}))^{-1}\,dt < \infty.$$

To normalize our nonconstant linear growth harmonic functions, we may assume that they all vanish at a fixed point $x \in M$. By the same argument as in the Proposition, we know that the lengths of the gradient of these linear growth harmonic functions are bounded subharmonic functions. We will utilize the following lemma concerning bounded subharmonic functions on manifolds with nonnegative Ricci curvature, which was proved by the first author in [2] (see Theorem 4).

Lemma. *Let M be a complete manifold with nonnegative Ricci curvature. If h is a bounded subharmonic function, then h satisfies*

$$\lim_{r\to\infty} V(B_x(r))^{-1} \int_{B_x(r)} h(y)\,dy = \sup_{y\in M} h(y).$$

Proof of Theorem. Let f be a linear growth harmonic on M. Applying the above lemma to the function $|\nabla f|^2$ we see that

$$\lim_{r\to\infty} V(B_x(r))^{-1} \int_{B_x(r)} |\nabla f|^2(y)\,dy = \sup_{y\in M} |\nabla f|^2(y) = \|\nabla f\|_\infty^2.$$

Let us now define the inner product on the space of linear growth harmonic functions which vanish at x by

$$\langle\langle \nabla f, \nabla g\rangle\rangle = \lim_{r\to\infty} V(B_x(r))^{-1} \int_{B_x(r)} \langle \nabla f, \nabla g\rangle.$$

The limit on the right-hand side always exists by the fact that both f, g and $f+g$ are linear growth harmonic functions and the polarization

$$2\langle \nabla f, \nabla g\rangle = |\nabla(f+g)|^2 - |\nabla f|^2 - |\nabla g|^2.$$

One checks readily that this is indeed an inner product since

$$\langle\langle \nabla f, \nabla f\rangle\rangle = \|\nabla f\|_\infty^2.$$

Let us now consider an s-dimensional subspace H of the space of linear growth harmonic functions. Let $\{f_i\}$, $1 \le i \le s$, be an orthonormal basis for H with respect to this inner product, that is, $\langle\langle \nabla f_i, \nabla f_j\rangle\rangle = \delta_{ij}$.

Consider the function

$$F(y) = \sum_{i=1}^{s} f_i^2(y),$$

which is invariant under orthogonal change of basis. Taking the Laplacian of F and integrating over the geodesic ball centered at x of radius r, we have

$$\begin{aligned} 2\int_{B_x(r)} \left(\sum_{i=1}^{s} |\nabla f_i|^2\right) &= \int_{B_x(r)} \Delta\left(\sum_{i=1}^{s} f_i^2\right) \\ &= \int_{\partial B_x(r)} \frac{\partial}{\partial r}\left(\sum_{i=1}^{s} f_i^2\right) \\ &\le \int_{\partial B_x(r)} \left|\nabla\left(\sum_{i=1}^{s} f_i^2\right)\right|. \end{aligned}$$

In order to estimate the integrand of the right-hand side at a point y_0, we consider the subspace

$$H_0 = \{f \in H | f(y_0) = 0\}.$$

H_0 must be of at most codimension-1 in H since the linear map L from H to R given by $L(f) = f(y_0)$ must have rank less than or equal to 1. Hence, by an orthogonal change of basis for H, we can write

$$F(y) = \sum_{i=1}^{s} \phi_i^2(y),$$

where $\phi_i(y_0) = 0$ for all $i \ne 1$. This implies that y_0 is a minimum point for each of the functions $\phi_i^2(y)$ for $i \ne 1$, so that

$$|\nabla F|(y_0) \le 2|\phi_1|(y_0)|\nabla\phi_1|(y_0).$$

However, since the ϕ_i's are orthonormal, we conclude that

$$\|\nabla\phi_1\|_\infty^2 = \langle\langle \nabla\phi_1, \nabla\phi_1\rangle\rangle = 1,$$

and therefore

$$|\phi_1(y_0)| \le r(y_0) \quad \text{and} \quad |\nabla F|(y_0) \le 2r(y_0),$$

where r is the Riemannian distance from x. This gives the estimate

$$\int_{B_x(r)} \left(\sum_{i=1}^{s} |\nabla f_i|^2\right) \le rA(\partial B_x(r)),$$

where $A(\partial B_x(r))$ denotes the area of the boundary of the geodesic ball of radius r centered at x. Dividing both sides by $V(B_x(r))$ and using the fact that

$$\langle\langle \nabla f_i, \nabla f_i\rangle\rangle = 1 \quad \text{for all } 1 \le i \le s,$$

we conclude that for any $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that for $r \ge R(\varepsilon)$

$$\frac{s-\varepsilon}{r} \le \frac{A(\partial B_x(r))}{V(B_x(r))}.$$

Integrating this inequality from $R = R(\varepsilon)$ to r, we have

$$V(B_x(r)) \ge \frac{V(B_x(R))}{R^{s-\varepsilon}} r^{s-\varepsilon}$$

for all $r \ge R$. Since ε is arbitrary, this establishes the theorem.

References

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